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AVOIDANCE IN ONE DIMENSION: A CONTINUOUS-MATRIX GAME

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ABSTRACT

The game is a two-person zero-sum game. On each play, each player selects any point on a line of finite length. The payoff is a trapezoidal function of the separation between the two selected points; it is constant for separations from zero to R_1, changes linearly between R_1 and R_2, and is zero for separations greater than R_2. The derivation and proof of the solution are interesting due to the discontinuities in the slope of the payoff function. The solution includes the special cases of triangular ($R_1 = 0$) and rectangular ($R_1 = R_2$) payoff functions. The game is related to search theory in its applicability to the barrier problem. Uniform distribution along the barrier is not in general an optimal strategy for either the maximizer (detector) or the avoider (transitor). In selecting optimal strategies the detector must have more information on the payoff function (lateral range curve) than is required by the transitor.

I. The Game

A solution is presented to the following two-person, zero-sum game:

Each player chooses any point on a line of definite length. The payoff is a single-valued, non-negative function of the distance along the line between the two points, having the shape of a trapezoid symmetrical about zero separation.

In analogy with the physical problem of a detection barrier, the payoff function will be called the lateral range curve (LRC), the maximizing player will be called the detector, and the minimizing player will be called the transitor. However, the game is more general in that, for instance, the payoff could just as well represent capture or damage, and the dimension of the line could just as well be time as distance. The game takes its name from the fact that the aim of the minimizer will always be to avoid the maximizer.

The optimal strategies which are presented as a solution to the game with a trapezoidal LRC will also be solutions for the special cases:

(1) definite range, or rectangular, LRC (upper and lower bases of trapezoid are equal) and (2) triangular LRC (upper base equal to zero). For the definite range LRC an alternate solution will also be given. Some special properties of the trapezoidal solution applied to the triangular LRC will be pointed out.

Although the rules of the game allow both players to know the LRC, it will be seen that, in order to apply the trapezoidal strategies, the transitor requires less information than does the detector. It is of course assumed that neither player knows the position chosen by his opponent on any one play.

II. Definitions of Symbols

The lateral range curve (LRC) is given by,

$$P(r) = C \text{ (constant)}, \quad r \le R_1,$$

$$P(r) = C \frac{R_2 - r}{R_2 - R_1}, \quad R_1 \le r \le R_2,$$

$$P(r) = 0 \quad , \quad r \geq R_2,$$

in which r is the separation between the transitor and the detector.

$$W = \int_{-R_2}^{+R_2} P(r)\, dr = C(R_1 + R_2).$$

In the detection problem analogy, W is the sweep width. A quantity that will appear in the solution is Z, which is defined as

$$Z = L \text{ Mod } \left(\frac{W}{C}\right) = L \text{ Mod } (R_1 + R_2),$$

in which L is the length of the line upon which the game is played. N is the integral number of times that w/c will go into L,

$$N = \frac{L-Z}{w/c} = \frac{L-Z}{R_1+R_2}$$

III. The Detector Strategy

The detector strategy for a trapezoidal LRC is a mixed strategy described by the following distribution of points. Measured from one end of the game line, there is a strategy point at y and at L-y for each value of y shown in table I. The frequency (probability of selection on each play) of each strategy point is given in table I opposite the corresponding value of y.

TABLE I

THE TRAPEZOIDAL DETECTOR STRATEGY

y	Frequency
R_1	$(N + 1)/(N + 1)(N + 2)$
$R_1 + W/C$	$N / (N + 1)(N + 2)$
$R_1 + 2\,W/C$	$(N - 1)/(N + 1)(N + 2)$
$R_1 + 3\,W/C$	$(N - 2)/(N + 1)(N + 2)$
.	.
.	.
.	.
$R_1 + NW/C$	$1 / (N + 1)(N + 2)$

Note that the largest value of y, $(R_1 + NW/C)$, will be greater than L if $R_1 > Z$. This is not a permissible strategy point since it is off of the game line. In this case a solution is obtained by letting the largest value of y be L instead of $R_1 + NW/C$, and letting the frequency of this point remain equal to $1/(N + 1)(N + 2)$

IV. The Transitor Strategy

The transitor strategy also consists of a set of points with corresponding frequencies. Measures from one end of the game line, there is a strategy point at X and at L - X for each value of X shown in table II. Again, the corresponding frequencies are shown opposite each value of X.

TABLE II

THE TRAPEZOIDAL TRANSITOR STRATEGY

X	Frequency
0	$(N + 1)/(N + 1)(N + 2)$
W/C	$N /(N + 1)(N + 2)$
$2W/C$	$(N - 1)/(N + 1)(N + 2)$
$3W/C$	$(N - 2)/(N + 1)(N + 2)$
.	.
.	.
.	.
NW/C	$1 /(N + 1)(N + 2)$

We see that the transitor requires less information about the LRC than does the detector to apply this strategy. The detector must know R_1 and R_2 whereas the transitor need only know their sum, $R_1 + R_2 = W/C$.

Since the transitor strategy is independent of R_1, we can show a completely general plot of the transitor trapezoidal strategy as a function of W/CL. In figure 1 the transitor strategy points are indicated by the solid lines. In the special case of $R_1 = 0$ (triangular LRC) the transitor and detector strategies are identical. The solid lines of figure 1 therefore also represent the detector strategy for the triangular LRC. The dashed lines and shaded areas in the figure will be discussed in a later section.

It is interesting to note that there are no discontinuities in the transitor trapezoidal strategy as W/C varies with respect to L. At each integral value of L/(W/C), pairs of strategy points coalesce and then diverge, and in this process the sum of the frequencies of the pair remains constant. This is illustrated by the frequencies shown in figure 1 on both sides of L/(W/C) equal to two, and to three.

V. The Game Value

The value of the game is given by

$$v = \frac{C}{N+1} \qquad \text{for } 0 \le Z \le 2R_1$$

$$v = \frac{C}{N+1} - \frac{(Z - 2R_1) \quad C}{(N+1)(N+2)(R_2 - R_1)} \quad \text{for } 2R_1 \le Z < R_1 + R_2$$

The value of Z must be less than $R_1 + R_2$ by definition. The game value is of course independent of the particular strategy used by either player. It is a unique function of C, R_1, R_2, and L, (Z and N are unique functions of R_1, R_2 and L). An optimal strategy is one which will prevent the opposing player from achieving an expected payoff better than the game value, whatever strategy the opposing player may use. The strategies for detector and transitor, presented in the previous two sections, are optimal, but they are not necessarily the only possible optimal strategies.

The game value is

(a) the minimum expected payoff for any transitor position if the detector uses an optimal strategy,

(b) the maximum expected payoff for any detector position if the transitor uses an optimal strategy, and therefore,

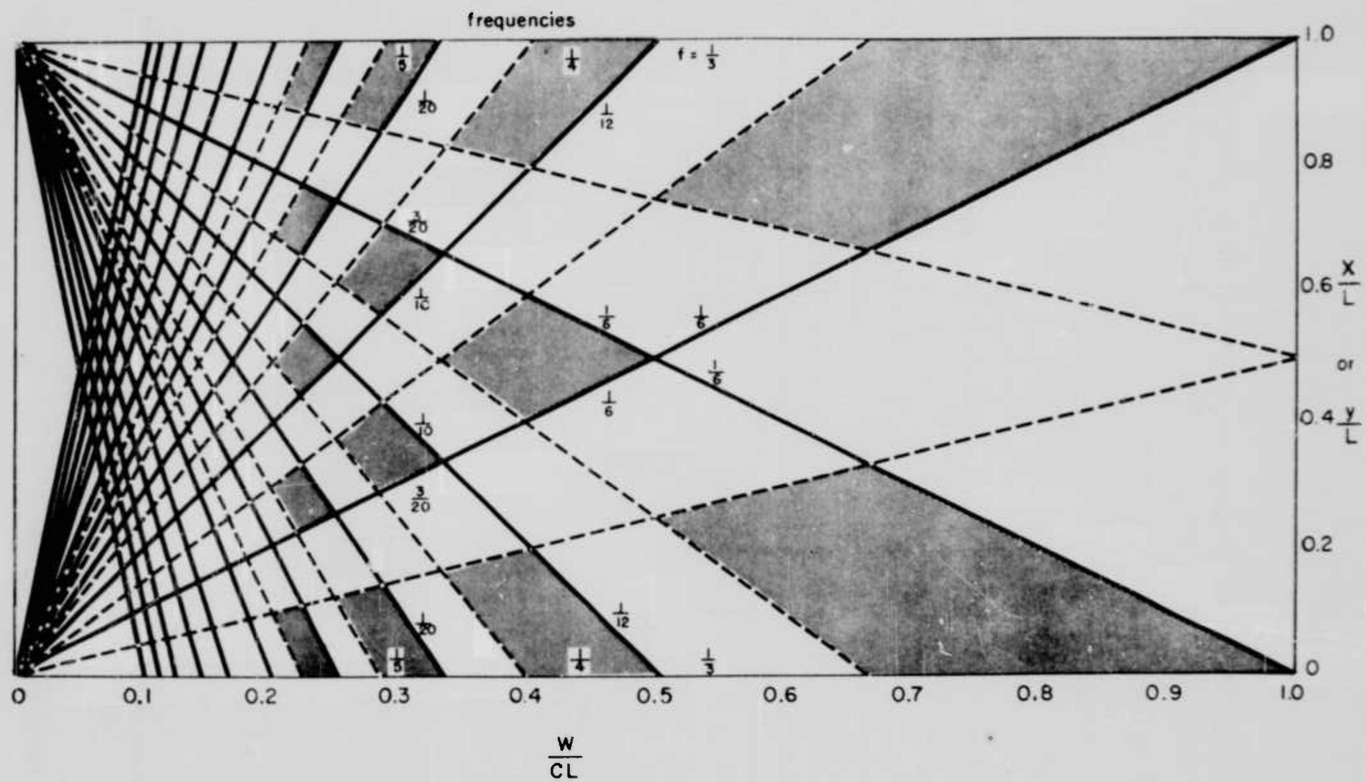

FIG. 1: GRAPHICAL REPRESENTATION OF SOME OPTIMAL STRATEGIES
Solid lines are transitor trapezoidal and detector triangular strategies.
Dashed lines are detector rectangular strategy.
Shaded areas are an alternate rectangular strategy for both.

(c) the expected payoff (i.e. the a priori detection probability on each play) if both players are using optimal strategies.

Here a priori means before the players have chosen their positions for the play. The choice of position is made by random selection from the distribution of positions represented by the optimal strategy.

VI. Special Cases of the Trapezoidal LRC

It has been pointed out that the optimal transitor and detector strategies given become identical for the special case of the triangular LRC, that is when the upper base of the trapezoid is zero. In this case, if either player uses this optimal strategy, then the expected payoff is independent of the position selected by the opposing player.

In the special case of the rectangular LRC, that is when the upper base of the trapezoid is equal to the lower base, the positions in the given optimal strategy for the detector become as shown by the dashed lines in figure 1. In this special case there is also another set of optimal strategies which have the same properties as the previously given strategies for the triangular LRC, namely:

(a) Identical strategies for transitor and detector, and

(b) Expected payoff independent of position for one player if other player uses this special rectangular strategy.

In this special rectangular strategy the player chooses with equal probability from among N + 1 channels. The channels are centered at

$$Z/2$$
$$Z/2 + W$$
$$Z/2 + 2W$$
$$\cdots$$
$$Z/2 + NW$$

and the channel widths are

$$\frac{W}{2} - \left| Z - \frac{W}{2} \right| .$$

Within a channel the choice of position is uniformly distributed over the width of the channel.

These channels are indicated by the shaded areas in figure 1.

VII. Method of Derivation of Solutions

Since there is no known algorithm (at least none known to the author) for solving this continuous-matrix game, the method by which these solutions were obtained should be of interest.

a The apparently simplest case, the rectangular LRC, was tired first. This was done by substituting a discrete matrix for the continuous matrix, that is by restricting the players to n discrete points along the line. By letting the n frequencies for one player be unknowns and by setting the n expected payoffs (corresponding to the n positions of the second player) equal to an unknown constant, we have n linear equations and n + 1 unknowns. An additional linear equation is given by the fact that the sum of the n frequencies is unity. In this case of the rectangular LRC this set of simultaneous linear equations yields a unique solution for the frequencies and for the unknown constant, which is the game value. Since the matrix is symmetric, the solution frequencies represent an optimal strategy for the other player as well. A solution was found in this way for a number of different values of n and from these solutions the continuous-matrix solution was induced. The solution so obtained is the one given in section VI of this paper. It has been tested for a number of cases of the rectangular LRC in a continuous matrix, but no general proof of this solution has been done.

b Next the triangular LRC was solved by the same method. Again the simultaneous linear equations could be solved to give the game value and identical optimal strategies for both players. These were the strategies shown in section III (for $R_1 = 0$) and in section IV.

c. The general case of the trapezoidal LRC (general in the sense that it includes the rectangular and triangular LRC's as special cases) does not yield a solution by this method. The solution was found by induction from the two special cases. Graphs of game value as a function of W/L for the rectangular and triangular LRC's suggested that the trapezoidal game value would be the one given in section V. It seemed of interest to see how the players would fare if they employed the triangular strategy (expressed in terms of W) against a trapezoidal LRC, giving them the benefit of using the W corresponding to the trapezoid. For the transitor it was found in a number of sample cases that he could by this means achieve a payoff function with a maximum equal to the suggested game value. The detector however did not achieve a payoff function minimum equal to this value. It was apparent that for the detector there is no advantage in going closer than R_1 to the end of the game line, so the triangular strategy was altered by moving all positions a distance R_1 from that end of the line from which they were measured. All positions were moved, rather than just the end ones, in order to preserve the spacing W/C, which seemed to be common to all solutions. Frequencies were kept unaltered. A number of sample cases showed that this altered strategy for the detector gave him a payoff function minimum equal to the suggested game value, which showed that this was the true game value for these sample cases, inasmuch as the transitor strategy gave the same value for his payoff function maximum.

VIII. General Proof of the Trapezoidal Solution

The sample cases of course did not constitute a general proof. The proof was accomplished only by considering all possible values of R_1, R_2, and L. The full proof will not be given here, but the method will be fully specified.

It is possible to replace the continuous game matrix by an infinite discrete matrix. In testing the postulated transitor strategy, for example, the columns of the infinite discrete matrix consist of the discrete transitor strategy points. The matrix is infinite because the value of L has so far been left open. It will be seen that any finite matrix will be a special case of the infinite matrix for which the proof is obtained. There is still a continuum of rows, because the expected payoff must now be determined for all possible detector positions to show that the maximum is equal to the postulated game value. But this can also be reduced to an infinite number of discrete positions by selecting only those detector positions at which discontinuities occur in the slope of the expected payoff as a function of the detector position. Then, because of the linearity of the payoff function between slope discontinuities, the extremum of the payoff at these discrete detector positions will also be the extremum of the payoff for all detector positions. The determination of the discrete detector positions is illustrated in table III. The X_i are the transitor strategy

TABLE III

i	=	1	2	3	4	5	etc.
X_i	=	0	Z	$R_1 + R_2$	$Z + R_1 + R_2$	$2R_1 + 2R_2$	...
$y = X_i - R_2$	=	(0)	$Z - R_2$	R_1	$Z + R_1$	$2R_1 + R_2$	...
$y = X_i - R_1$	=	(0)	$Z - R_1$	R_2	$Z + R_2$	$R_1 + 2R_2$	...
$y = X_i + R_1$	=	R_1	$Z + R_1$	$2R_1 + R_2$	$Z + 2R_1 + R_2$	$3R_1 + 2R_2$	...
$y = X_i + R_2$	=	R_2	$Z + R_2$	$R_1 + 2R_2$	$Z + R_1 + 2R_2$	$2R_1 + 3R_2$	...

points starting from one end of the line. Notice that $Z = L - NW/C$ is the last strategy point measured from the other end of the line as the strategy is defined in section IV. This point is bound to lie between X_1 and X_3 because $0 \leq Z < W/C = R_1 + R_2$ by definition (section II). Discontinuities in the payoff slope will occur at detector positions, y, given by $X_i \pm R_1$ and $X_i \pm R_2$ for all values of X_i because R_1 and R_2 are the X-y separations at which the trapezoidal payoff function has slope discontinuities. Note that for i = 1, $X_i - R_1$ and $X_i - R_2$ are negative which is not allowed so the discontinuity occures at y = 0.

Note also that for $i > 3$ we have $y(X_i) = y(X_{i-2}) + R_1 + R_2$. We therefore have determined all of the discrete values of y without carrying the table to higher values of i. The resulting infinite discrete matrix is shown in table IV. Frequencies of the transitor strategy points are indicated by f_i. The elements

TABLE IV

THE INFINITE DISCRETE MATRIX

$i =$	1	2	3	4	5	6	etc.
$(N+1)(N+2)\ f_i =$	$N+1$	1	N	2	$N-1$	3	...
$X_i =$	0	Z	R_1+R_2	R_1+R_2+Z	$2R_1+2R_2$	$2R_1+2R_2+Z$	
$y = 0$	0	Z					
$Z-R_2$	$Z-R_2$	R_2					
$Z-R_1$	$Z-R_1$	R_1	$2R_1+R_2-Z$				
R_1	R_1	$Z-R_1$	R_2				
$Z+R_1$	$Z+R_1$	R_1	$Z-R_2$	R_2			
R_2	R_2	$Z-R_2$	R_1	$Z+R_1$			
$Z+R_2$		R_2	$Z-R_1$	R_1	$2R_1+R_2-Z$		
$2R_1+R_2$		$2R_1+R_2-Z$	R_1	$Z-R_1$	R_2		
$Z+2R_1+R_2$			$Z+R_1$	R_1	$Z-R_2$	R_2	
Etc.	...	...	...	...	...	...	...

of the matrix are not payoff values but are values of $r_i = X_i - y$, which are the separations which determine the payoff values, $P(r_i)$. Separations greater than R_2 have been omitted because we know that for such separations the payoff value is zero. Now it only remains to find the expected payoff,

$$\langle P \rangle = \sum_{i=1}^{\infty} f_i\, P(r_i),$$

for each value of y.

In this we make use of the relationships,

$$f_i + f_{i+1} = 1/(N+1) \quad , \quad i \text{ odd}$$
$$f_i + f_{i+1} = 1/(N+2) \quad , \quad i \text{ even}$$
$$f_i - f_{i+2} = 1/(N+1)(N+2) \quad , \quad i \text{ odd}$$
$$f_i - f_{i+2} = -1/(N+1)(N+2) \, , \quad i \text{ even,}$$

by means of which, it turns out, the frequencies can be factored out of $\langle P \rangle$ for all values of y.

Furthermore, we find that each value of y greater than those shown in table IV produces an expected payoff identical to that for one of the y's in table IV. We have already seen, in connection with the discussion of table III, that each higher value of y, say y_b, will be equal to one of the given values, say y_a, plus an integral multiple of $(R_1 + R_2)$. Thus for each separation r_i associated with y_a, there will be an identical r_j associated with y_b, with X_j being equal to X_i plus the same integral multiple of $(R_1 + R_2)$.

$$y_b = y_a + n(R_1 + R_2)$$

$$x_j = x_i + n(R_1 + R_2)$$

$$r_j = x_j - y_b = x_i - y_a = r_i \ .$$

See, for example, $y = Z + R_1$ and $y = Z + 2R_1 + R_2$ in table IV. And we see from table III that $j = i + 2n$. So, for y_b,

$$\langle P_b \rangle = \sum_{j=1}^{\infty} = f_j P(r_j)$$

$$= \sum_{j=1}^{2n} f_j P(r_j) + \sum_{j=1+2n}^{\infty} f_j P(r_j) \ .$$

Changing indices from j to i and recalling that $P(r_j) = P(r_i)$ since $r_j = r_i$, we have,

$$\langle P_b \rangle = \sum_{i=1-2n}^{0} f_{i+2n} P(r_i) + \sum_{i=1}^{\infty} f_{i+2n} P(r_i)$$

The first summation is zero because $P(r_i) = 0$ for $i \leq 0$ (if $y_a \geq R_2$, which means we must use such y_a in generating $\langle P_b \rangle$). The second summation is identical to $\langle P_a \rangle$ except that all indices of f have increased by 2n, an even number. Thus all indices retain their evenness or oddness. Therefore,

$$\langle P_b \rangle = \langle P_a \rangle \ .$$

The values of $y \geq R_2'$ in table IV are sufficient to generate $<P>$ for all higher values of y. Therefore, if we find the maximum $<P>$ for the values of y in table IV, this will also be the maximum for all y in the infinite discrete matrix, and since the expected payoff is a linear function of y between the discrete values of y, the maximum so obtained is the maximum for all values in the y continuum.

In finding $<P>$ for each value of y in table IV, it is necessary to consider various different limits on the variables due to the slope discontinuities in P(r). As a single example, in y = 0 the value of P(Z) depends upon whether $0 \leq Z \leq R_1$ or $R_1 \leq Z \leq R_2$ or $Z > R_2$. In all, there are five different regions of the variables and for each of these five regions it is necessary to find the maximum among the $<P>$ for all of the y in table IV. The five regions are shown in figure 2.

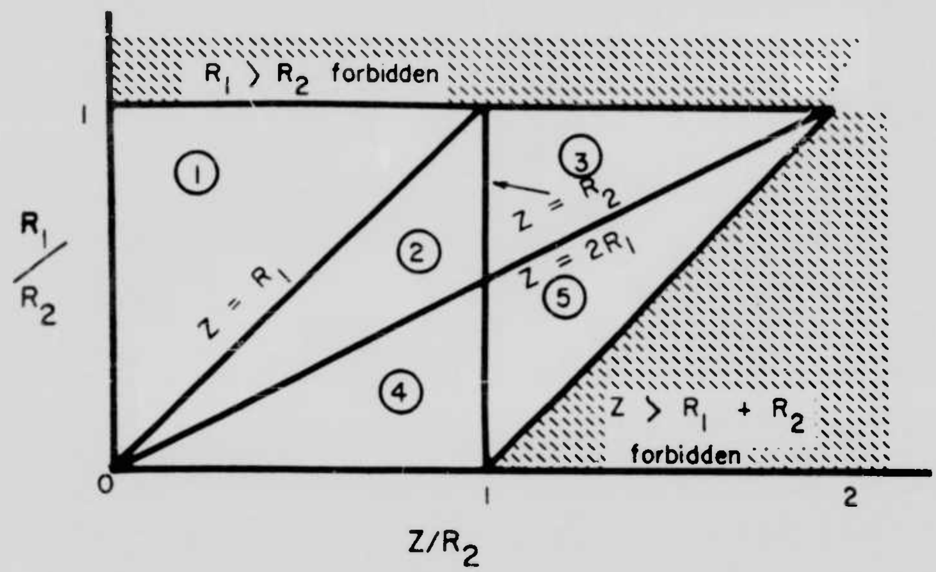

FIG. 2: REGIONS OF VARIABLES IN TESTING TRANSITOR STRATEGY

In each region the maximum $<P>$ was found to be equal to the game value postualted in section V,

$$<P>_{max} = \frac{C}{N+1} \text{ in regions 1, 2, and 3}$$

$$<P>_{max} = \frac{C}{N+1} - \frac{(Z-2R_1)C}{(N+1)(N+2)(R_2-R_1)} \text{ in 4 and 5.}$$

These five regions cover all possible values of Z, R_1, and R_2, and therefore cover all possible values of R_1, R_2, and L.

To complete the proof we must show that

$$\langle P \rangle_{min} = v$$

for all possible transitor positions with the detector following the optimal strategy given in section III. This is done in the same manner, but is slightly more complicated in that:

(a) The last strategy point as measured from one end of the game line does not always lie between the first two measured from the other end.

(b) The last strategy point as measured from one end of the game line is not always a distance $R_1 + R_2$ from the next to last.

(c) There are eight regions of the variables in which $\langle P \rangle$ must be determined for each discrete value of x. These regions are shown in figure 3.

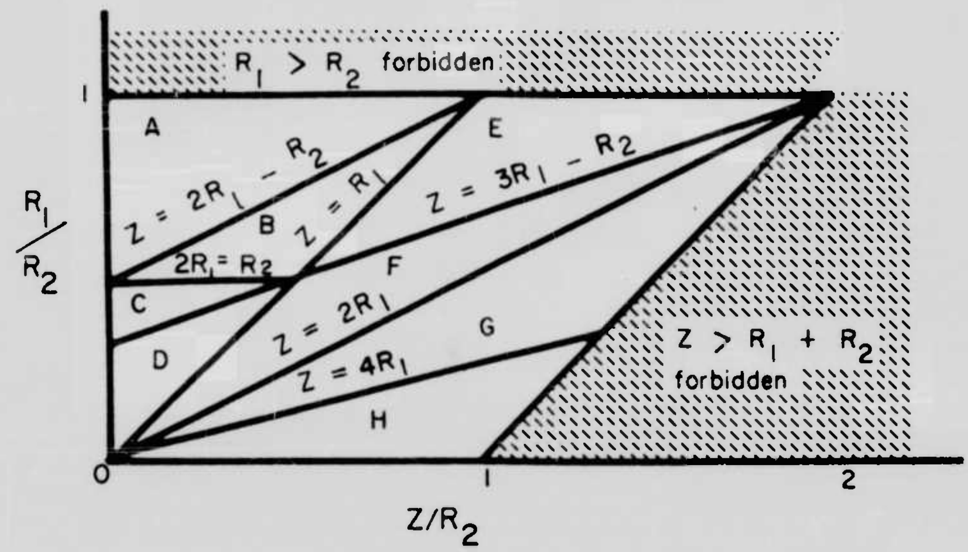

FIG. 3: REGIONS OF VARIABLES IN TESTING DETECTOR STRATEGY

Following the same procedure as was described for the test of the transitor strategy, it is found that among all transitor positions:

$$\langle P \rangle_{min} = \frac{C}{N+1} \quad \text{in regions A, B, C, D, E, and F}$$

$$\langle P \rangle_{min} = \frac{C}{N+1} - \frac{(Z-2R_1)C}{(N+1)(N+2)(R_2-R_1)} \quad \text{for G and H,}$$

which completes the proof.